ON STEADY THIRD GRADE FLUIDS EQUATIONS

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ABSTRACT. Let Ω be a simply connected, bounded, smooth domain of $\mathbb{R}^2$ or $\mathbb{R}^3$. We consider the equation of steady motion of a third grade fluid in Ω with homogeneous Dirichlet boundary conditions. We prove that the monotonicity technique used by Paicu [17] in the full space for unsteady motion allows to obtain the existence of a $W_0^{1,4}$ solution provided that the forcing belongs to $W^{-1,\frac{4}{3}}$. The size of the forcing is arbitrary.

KEY WORDS: Non-Newtonian fluids, third grade fluids.

1. INTRODUCTION

We consider in this paper the equations of steady motion of a third grade fluid in a bounded domain Ω endowed with homogeneous Dirichlet boundary conditions:

$$
\begin{aligned}
-\nu\triangle u + u\cdot\nabla u - \alpha_2 \operatorname{div}(A^2) - \alpha_1 \operatorname{div}(u\cdot\nabla A + L^t A + AL) - \beta \operatorname{div}(|A|^2 A) &= f - \nabla p, && \text{in } \Omega,\\
\operatorname{div} u &= 0, && \text{in } \Omega,\\
u &= 0, && \text{on } \partial\Omega.
\end{aligned}
\tag{1}
$$

Above, $u(x)$ denotes the velocity vector field, p is the fluid pressure, L denotes the gradient matrix of the velocity $L = \nabla u = (\partial_j u_i)_{i,j}$, $A = L + L^t$ and $\nu, \alpha_1, \alpha_2, \beta$ are some material constants that, according to the thermodynamic study performed by Fosdick and Rajagopal [14], must verify the following assumptions:

$$\nu \geq 0, \quad \alpha_1 \geq 0, \quad \beta \geq 0 \quad \text{and} \quad |\alpha_1 + \alpha_2| \leq (24\nu\beta)^{1/2}.$$

We refer to [8] for further details concerning the modeling and physical significance of these equations. We will assume that Ω is a bounded, smooth, simply-connected domain of $\mathbb{R}^2$ or $\mathbb{R}^3$. The unsteady version of the above equations involves the additional term $\partial_t(u - \alpha_1\triangle u)$ on the left-hand side.

The first mathematical results on third grade fluids are due to [1, 2], see also [6, 18]. These articles treat the unsteady case, assume that the initial data belongs to H^3 or $W^{2,r}$ with $r > 3$ and prove global existence and uniqueness of solutions for small initial data or local existence and uniqueness of solutions for large initial data.

Recently, in the case of the full space $\mathbb{R}^2$ and $\mathbb{R}^3$, two of the authors proved the global existence of solutions for large H^2 initial data, and also the uniqueness of such solutions in dimension two, see [7, 8]. These results were extended in [9] to bounded domains if Navier slip boundary conditions are imposed, see also [10] for the case of second grade fluids.

Very recently, one of the authors went even further and was able to prove in [17] the global existence of solutions in the case of the full space for large initial data in H^1. This requires a new idea, even though there is a well-known a priori energy estimate in H^1. Indeed, it seems to be very hard to pass to the limit in an approximation procedure with compactness methods since the *a priori* estimates we have give control over the $W^{1,4}$ norm of the solution at most and we would need to pass to the limit in terms like $\operatorname{div}(|A|^2A)$. The new idea of [17] is the use of the monotonicity of some of the non-linear part of the equation in order to pass to the limit. Let us note that this monotonicity method was already used for example by J.-L. Lions [16, pages 155-162] for a simpler model related to our equation. We would also like to point out that using this approach it is possible to construct a unique L^2 solution in the particular case where $\alpha_1 = 0$, see [15].

As far as the steady case is concerned, Bernard and Ouazar [3] were able to extend the results available for second grade fluids (see [5, 12]) to the case of third grade fluids. They prove the existence and uniqueness of an H^3 solution provided that the forcing f is small and belongs to H^1.

In this paper we use the monotonicity idea of [17] to prove existence of solutions in the steady state case for bounded domains with homogeneous Dirichlet boundary conditions. We are then able to remove the smallness assumption of [3] and moreover consider less regular forcing. More precisely, we prove that for every forcing $f \in W^{-1,\frac{4}{3}}$, there exists a $W^{1,4}$ solution of the steady third grade fluids equations. We also prove that every such weak solution verifies an energy equality. A special smoothing procedure must be used for the proof to work. We also note that the restrictions that we impose on the coefficients due to the monotonicity argument are weaker than those of [17].

Theorem 1 (H^1 solutions)**.** *Assume that $f \in W^{-1,\frac{4}{3}}$ and that the material coefficients verify the following conditions: $\nu \geq 0$, $\beta > 0$ and $|\alpha_1| \leq \sqrt{8\beta\nu}$ in dimension two and $\nu \geq 0$, $\beta > 0$ and $3\alpha_1^2 + 4(\alpha_1 + \alpha_2)^2 \leq 24\nu\beta$ in dimension three. There exists a solution $u \in W_0^{1,4}$ of equation* (1) *in the sense of distributions. In addition, the following energy equality holds true:*

$$\nu \int_\Omega |A|^2 + \beta \int_\Omega |A|^4 + (\alpha_1 + \alpha_2) \int_\Omega \operatorname{tr}(A^3) = 2\langle f, u \rangle_{W^{-1,\frac{4}{3}}, W_0^{1,4}}. \tag{2}$$

In the theorem above, we extended the result of [17] to the steady case and to bounded domains. One might wonder if the theory of H^2 solutions of [8] cannot be extended likewise. The difficulty when one considers the Dirichlet condition and performs H^2 energy estimates arises from the term $-\int_\Omega \nabla p \cdot \triangle u$. This term is of lower order for Navier boundary condition (and this allows to prove the results of [9]), but does not seem to be of lower order for Dirichlet boundary condition.

We conclude this work with a weak-strong uniqueness result. When the forcing f is small and belongs to H^1, one can construct a small H^3 solution as in [3], but also $W^{1,4}$ solutions by Theorem 1. We prove that the $W^{1,4}$ and (the small) H^3 solutions must necessarily be equal.

Theorem 2 (weak-strong uniqueness)**.** *Assume the same hypothesis on the material coefficients as in Theorem 1 but with strict inequalities: $\nu > 0$, $\beta > 0$ and $|\alpha_1| < \sqrt{8\beta\nu}$ in dimension two and $\nu > 0$, $\beta > 0$ and $3\alpha_1^2 + 4(\alpha_1+\alpha_2)^2 < 24\nu\beta$ in dimension three. Let u and $\widetilde{u}$ be two solutions of Equation* (1) *belonging to $W_0^{1,4}$ which are associated to the same forcing. Let $p = 3$ if the space dimension is three and $p > 2$ arbitrary in dimension two. There exists a constant M depending only on the material coefficients, the domain Ω and the constant p such that if $\widetilde{u} \in W^{2,p}$ and $\|\widetilde{u}\|_{W^{2,p}} \leq M$, then $u = \widetilde{u}$.*

2. Notation and preliminary results

For two vector fields u and $\widetilde{u}$ we define the scalar product $u \cdot \widetilde{u} = \sum_i u_i \widetilde{u}_i$ and $|u| = (u \cdot u)^{\frac{1}{2}}$. For two matrices A and B we set $A : B = \sum_{ij} A_{ij} B_{ij}$ and $|A| = (A : A)^{\frac{1}{2}}$.

Let Ω be a bounded, smooth and simply connected domain of $\mathbb{R}^n$, $n = 2, 3$. We denote by $W_0^{m,p}$ the closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$ and by $W_{0,\sigma}^{m,p}$ the space of divergence free vector fields belonging to $W_0^{m,p}(\Omega)$. We will also use the classical notation $H_0^m = W_0^{m,2}(\Omega)$. We denote by V the space of divergence free vector fields in H_0^1. The spaces H^{-1}, respectively $W^{-1,\frac{4}{3}}$, denote the dual spaces of H_0^1, respectively $W_0^{1,4}$. For f and g in two dual spaces X and X' we denote by $\langle f, g \rangle_{X,X'}$ the usual duality parenthesis. If f and g are vector fields then $\langle f, g \rangle_{X,X'} = \sum_i \langle f_i, g_i \rangle_{X,X'}$ and we will also use a similar notation for matrices. We will use this notation mainly for $X = L^2$, $X = W_0^{1,4}$ or $X = \mathcal{D} \equiv C_0^\infty(\Omega)$. It is clear that $\langle f, g \rangle_{L^2} = \langle f, g \rangle_{W^{-1,\frac{4}{3}}, W_0^{1,4}} = \langle f, g \rangle_{\mathcal{D}', \mathcal{D}}$ whenever these expressions make sense. For this reason, we will sometimes drop the index ${}_{X,X'}$ when no confusion can occur.

We observe now that equation (1) makes sense in the distributions if $u \in W_0^{1,3}$. Indeed, the terms $A^2, L^t A, AL$ and $|A|^2 A$ all belong to L^1, so the divergence of these terms is well-defined in the sense of distributions. Next, the terms $\triangle u$ and $u \cdot \nabla u$ clearly make sense in the space of distributions. Finally, we observe that $\operatorname{div}(u \cdot \nabla A) = \sum_i \partial_i \operatorname{div}(u_i A)$. Since $u_i A \in L^1$, the expression $\sum_i \partial_i \operatorname{div}(u_i A)$ is well defined in the sense of distributions, and so is $\operatorname{div}(u \cdot \nabla A)$.

We next introduce the following smoothing operator

$$J_\varepsilon f = A_\varepsilon B_\varepsilon f,$$

where B_ε is a cut-off operator at distance $\geq 2\varepsilon$ of the boundary and A_ε is the usual smoothing by convolution operator. More precisely, let ϕ be an even function belonging to $C_0^\infty(B(0,1))$, $\phi \geq$

$0, \int \phi = 1$; we set

$$A_\varepsilon f = \phi_\varepsilon * f$$

where $\phi_\varepsilon = \frac{1}{\varepsilon^n}\phi\big(\frac{x}{\varepsilon}\big)$ and $n \in \{2,3\}$ is the space dimension. Next, for ε small enough there exist a function $h_\varepsilon \in C_0^\infty(\Omega;[0,1])$ such that $h_\varepsilon(x) = 0$ if $d(x,\partial\Omega) \le 2\varepsilon$, and $h_\varepsilon(x) = 1$ if $d(x,\partial\Omega) \ge 3\varepsilon$. We can further assume that $\|\nabla^k h_\varepsilon\|_{L^\infty(\Omega)} = O(1/\varepsilon^k)$. We define then B_ε to be the operator of multiplication by the cut-off function h_ε:

$$B_\varepsilon f = h_\varepsilon f.$$

We recall that if $f \in W_0^{m,p}(\Omega)$ then

$$J_\varepsilon f \longrightarrow f \quad \text{in} \quad W^{m,p}(\Omega)$$

and $J_\varepsilon f \in C_0^\infty(\Omega)$.

Let $w \in W_{0,\sigma}^{m,p}(\Omega)$. In dimension 2, since Ω is simply connected, it is well-known that there exists a uniquely defined stream function $\psi \in W_0^{m+1,p}(\Omega)$, i.e. $\nabla^\perp\psi = w$. In dimension 3, according to [4, Theorem 2.1] and using again that Ω is a simply connected we deduce that there exists a vector field $\psi \in W_0^{m+1,p}(\Omega)^3$ such that $w = \operatorname{curl}\psi$. The vector field ψ is not necessarily uniquely defined; however there exists a linear continuous operator

$$S = S(\Omega,m,p) : W_{0,\sigma}^{m,p}(\Omega) \longrightarrow W_0^{m+1,p}(\Omega)^3$$

such that $\operatorname{curl} Sw = w, \forall w \in W_{0,\sigma}^{m,p}(\Omega)$.

We will denote in the following by ψ either the uniquely defined stream function mentioned above if $n = 2$ or the vector field Sw if $n = 3$. Observe that the mapping $u \mapsto \psi$ is linear and continuous from $W_{0,\sigma}^{m,p}(\Omega)$ to $W_0^{m+1,p}(\Omega)^{2n-3}$. We introduce now the following smoothing operator for divergence free vector fields. For $w \in W_{0,\sigma}^{m,p}(\Omega)$ we set

$$\widetilde{J}_\varepsilon w = \nabla^\perp(J_\varepsilon\psi) \qquad \text{if } n = 2$$

and

$$\widetilde{J}_\varepsilon w = \operatorname{curl}(J_\varepsilon\psi) \qquad \text{if } n = 3.$$

We observe that in both cases $\widetilde{J}_\varepsilon w$ is a divergence free C_0^∞ vector field such that

$$\widetilde{J}_\varepsilon w \longrightarrow w \quad \text{in } W^{m,p}(\Omega).$$

We will later use the following classical Hardy inequality.

Lemma 3. *If $f \in W_0^{m,p}(\Omega)$ then $f(x)/d^m(x,\partial\Omega) \in L^p(\Omega)$ and*

$$\Big\|\frac{f(x)}{d^m(x,\partial\Omega)}\Big\|_{L^p(\Omega)} \le C\|f\|_{W^{m,p}(\Omega)}.$$

Let us introduce the following non-linear operator $\mathcal{R} : W_0^{1,4} \to W^{-1,\frac43}$ defined by

(3) $$W_0^{1,4} \ni u \mapsto \mathcal{R}(u) = -\nu\triangle u - \alpha_1 \operatorname{div}(L^tA + AL) - \alpha_2 \operatorname{div}(A^2) - \beta \operatorname{div}(|A|^2A) \in W^{-1,\frac43}.$$

We skip the trivial verification that $\mathcal{R}(u)$ indeed belongs to $W^{-1,\frac43}$ if $u \in W_0^{1,4}$. The operator $\mathcal{R}$ verifies the following important monotonicity property:

Lemma 4. *Assume that the material coefficients verify the following conditions: $\nu,\beta \ge 0, |\alpha_1| \le \sqrt{8\beta\nu}$ in dimension two and $\nu,\beta \ge 0, 3\alpha_1^2 + 4(\alpha_1+\alpha_2)^2 \le 24\nu\beta$ in dimension three. Then the operator $\mathcal{R}$ is monotonic on $W_{0,\sigma}^{1,4}$:*

$$\langle \mathcal{R}(u) - \mathcal{R}(\widetilde{u}), u - \widetilde{u}\rangle_{W^{-1,\frac43},W_0^{1,4}} \ge 0$$

for all vector fields $u, \widetilde{u} \in W_{0,\sigma}^{1,4}$.

Proof. Let $u, \widetilde{u} \in W_{0,\sigma}^{1,4}$. We use the notation

$$A = A(u), \qquad \widetilde{A} = A(\widetilde{u}), \qquad L = L(u), \qquad \widetilde{L} = L(\widetilde{u}), \qquad w = u - \widetilde{u}.$$

Using that $\triangle u = \operatorname{div} A$, $\triangle \widetilde{u} = \operatorname{div} \widetilde{A}$ and integrating by parts we readily obtain that

$$\begin{aligned}
&\langle \mathcal{R}(u) - \mathcal{R}(\widetilde{u}), u - \widetilde{u} \rangle_{W^{-1,\frac{4}{3}}, W_0^{1,4}} \\
&\quad = \int_\Omega \big[\nu(A - \widetilde{A}) + \alpha_1 (L^t A + AL - \widetilde{L}^t \widetilde{A} - \widetilde{A}\widetilde{L}) + \alpha_2 (A^2 - \widetilde{A}^2) + \beta(|A|^2 A - |\widetilde{A}|^2 \widetilde{A}) \big] : (L - \widetilde{L}) \\
&\quad = \frac{1}{2} \int_\Omega \big[\nu(A - \widetilde{A}) + \alpha_1 (L^t A + AL - \widetilde{L}^t \widetilde{A} - \widetilde{A}\widetilde{L}) + \alpha_2 (A^2 - \widetilde{A}^2) + \beta(|A|^2 A - |\widetilde{A}|^2 \widetilde{A}) \big] : (A - \widetilde{A}).
\end{aligned}$$

where we used that the matrix between the square parenthesis is symmetric. Since the matrix $A - \widetilde{A}$ is symmetric, we next observe that

$$(L^t A + AL) : (A - \widetilde{A}) = L^t A : (A - \widetilde{A}) + AL : (A - \widetilde{A}) = 2AL : (A - \widetilde{A})$$

and

$$(\widetilde{L}^t \widetilde{A} + \widetilde{A}\widetilde{L}) : (A - \widetilde{A}) = 2\widetilde{A}\widetilde{L} : (A - \widetilde{A})$$

Using the identity

$$(|A|^2 A - |\widetilde{A}|^2 \widetilde{A}) : (A - \widetilde{A}) = \frac{1}{2}(|A|^2 - |\widetilde{A}|^2)^2 + \frac{1}{2}|A - \widetilde{A}|^2 (|A|^2 + |\widetilde{A}|^2)$$

we get that

$$\begin{aligned}
(4)\quad 2\langle \mathcal{R}(u) - \mathcal{R}(\widetilde{u}), u - \widetilde{u} \rangle_{W^{-1,\frac{4}{3}}, W_0^{1,4}} &= \int_\Omega \Big[\nu |A - \widetilde{A}|^2 + 2\alpha_1 (AL - \widetilde{A}\widetilde{L}) : (A - \widetilde{A}) \\
&\quad + \alpha_2 (A^2 - \widetilde{A}^2) : (A - \widetilde{A}) + \frac{\beta}{2}(|A|^2 - |\widetilde{A}|^2)^2 + \frac{\beta}{2}|A - \widetilde{A}|^2 (|A|^2 + |\widetilde{A}|^2) \Big].
\end{aligned}$$

Let $M = L - \widetilde{L}$, $C = A - \widetilde{A}$ and $B = A + \widetilde{A}$. We observe that B and C are trace free symmetric matrices and $C = M + M^t$. Recalling that $\operatorname{tr}(A_1 A_2) = \operatorname{tr}(A_2 A_1) = A_1 : A_2$ for any $n \times n$ matrices A_1 and A_2 we first observe that

$$(A^2 - \widetilde{A}^2) : (A - \widetilde{A}) = \operatorname{tr}\big[(A^2 - \widetilde{A}^2)(A - \widetilde{A})\big] = \operatorname{tr}\big[(A - \widetilde{A})^2 (A + \widetilde{A})\big] = C^2 : B.$$

Next, using that $A = \frac{B+C}{2}$ and $\widetilde{A} = \frac{B-C}{2}$ one has that

$$(AL - \widetilde{A}\widetilde{L}) : (A - \widetilde{A}) = \frac{1}{2}[(B + C)L - (B - C)\widetilde{L}] : C = \frac{1}{2}(BM) : C + \frac{1}{2}(CL + C\widetilde{L}) : C.$$

But $(CL) : C = L : C^2 = \frac{1}{2}(L + L^t) : C^2 = \frac{1}{2}A : C^2$ and similarly $(C\widetilde{L}) : C = \frac{1}{2}\widetilde{A} : C^2$, so

$$(AL - \widetilde{A}\widetilde{L}) : (A - \widetilde{A}) = \frac{1}{2}B : (CM^t) + \frac{1}{4}B : C^2 = \frac{1}{4}B : (CM^t + MC) + \frac{1}{4}B : C^2.$$

Furthermore, replacing $A = \frac{B+C}{2}$ and $\widetilde{A} = \frac{B-C}{2}$ in (4) we finally observe that

$$\begin{aligned}
&2\langle \mathcal{R}(u) - \mathcal{R}(\widetilde{u}), u - \widetilde{u} \rangle_{W^{-1,\frac{4}{3}}, W_0^{1,4}} \\
(5)\quad &= \int_\Omega \Big[\nu |C|^2 + \frac{\beta}{2}(C : B)^2 + \frac{\beta}{4}|C|^4 + \frac{\beta}{4}|C|^2 |B|^2 + \big(\alpha_2 + \frac{\alpha_1}{2}\big) C^2 : B + \frac{\alpha_1}{2} E : B \Big], \\
&\equiv \int_\Omega F(B)
\end{aligned}$$

where

$$E = CM^t + MC$$

is a symmetric matrix. Let us fix C, M and determine the minimum of the functional $F(B)$. We identify $\mathcal{M}_3(\mathbb{R})$ with $\mathbb{R}^9$, we consider F to be defined on $\mathcal{M}_3(\mathbb{R})$ by the same formula and we compute the minimum of F constrained by $g(B) \equiv \operatorname{tr}(B) = 0$. We do not include the constraint that B is symmetric since it will turn out that the minimum point of F is symmetric; so the constraint that B is symmetric is unnecessary. It is clear from the formula of $F(B)$ that $F(B) \to \infty$ if $|B| \to \infty$. Therefore, there exists a global minimum point of F on $\mathcal{M}_3(\mathbb{R})$ constrained by $g = 0$. Let B_0 be such a global minimum point. By the Lagrange multipliers method, one has that $\nabla F(B_0)$ must be proportional to $\nabla g(B_0)$. Clearly,

$$\nabla F(B_0) = \beta(C : B_0)C + \frac{\beta}{2}|C|^2 B_0 + \big(\alpha_2 + \frac{\alpha_1}{2}\big)C^2 + \frac{\alpha_1}{2}E \quad \text{and} \quad \nabla g(B_0) = I$$

so there exists $\lambda \in \mathbb{R}$ such that

$$\beta(C : B_0)C + \frac{\beta}{2}|C|^2 B_0 + \big(\alpha_2 + \frac{\alpha_1}{2}\big)C^2 + \frac{\alpha_1}{2}E = \lambda I. \tag{6}$$

Taking the trace of the above relation and observing that

$$\operatorname{tr}(E) = \operatorname{tr}(CM^t + MC) = \operatorname{tr}(M^t C + MC) = \operatorname{tr}(C^2) = |C|^2 \tag{7}$$

we get that

$$\lambda = \frac{\alpha_1 + \alpha_2}{n}|C|^2. \tag{8}$$

We next take the scalar product of (6) with C and use that

$$E : C = (CM^t) : C + (MC) : C = M^t : C^2 + M : C^2 = C : C^2 = \operatorname{tr}(C^3) \tag{9}$$

to obtain

$$C : B_0 = -\frac{2(\alpha_1 + \alpha_2)\operatorname{tr}(C^3)}{3\beta|C|^2}. \tag{10}$$

Plugging (8) and (10) in (6) allows to deduce the value of B_0:

$$B_0 = \frac{4(\alpha_1 + \alpha_2)\operatorname{tr}(C^3)}{3\beta|C|^4}C - \frac{(\alpha_1 + 2\alpha_2)}{\beta|C|^2}C^2 - \frac{\alpha_1}{\beta|C|^2}E + \frac{2(\alpha_1 + \alpha_2)}{n\beta}I. \tag{11}$$

We observe first that, as claimed above, B_0 is indeed a symmetric matrix. Next, we observe that $|C^2|^2 = |C|^4/2$; this follows from a trivial computation if the matrix C is trace-free and diagonal. The general case follows by diagonalising the matrix $C = O\overline{C}O^{-1}$ with O an orthogonal matrix and $\overline{C}$ a diagonal trace free matrix and writing $|C^2|^2 = \operatorname{tr}(C^4) = \operatorname{tr}(O\overline{C}^4 O^{-1}) = \operatorname{tr}(\overline{C}^4) = [\operatorname{tr}(\overline{C}^2)]^2/2 = [\operatorname{tr}(C^2)]^2/2 = |C|^4/2$. Moreover, it is easy to obtain as above that

$$E : C^2 = C^2 : C^2 = |C|^4/2. \tag{12}$$

Using these observations we can compute $C^2 : B_0$ and find

$$C^2 : B_0 = \frac{4(\alpha_1 + \alpha_2)[\operatorname{tr}(C^3)]^2}{3\beta|C|^4} + \frac{\alpha_1 + \alpha_2}{\beta}\big(\frac{2}{n} - 1\big)|C|^2. \tag{13}$$

We next express $E : B_0$. This term contains $|E|^2$ that we need to compute. Let $D = M - M^t$. It is a simple calculation to check that

$$E = C^2 + \frac{DC - CD}{2}.$$

We also observe that the matrices C^2 and $DC - CD$ are perpendicular with respect to the scalar product of matrices. Indeed, $C^2 : (DC) = C^2 : (CD) = C^3 : D$. Therefore

$$|E|^2 = |C^2|^2 + \frac{|CD - DC|^2}{4} = \frac{|C|^4}{2} + \frac{|CD - DC|^2}{4} \tag{14}$$

Taking the scalar product of (11) with E and using relations (7), (9), (12) and (14) we obtain after a few calculations that

$$E : B_0 = \frac{4(\alpha_1 + \alpha_2)[\operatorname{tr}(C^3)]^2}{3\beta|C|^4} + \frac{\alpha_1 + \alpha_2}{\beta}\big(\frac{2}{n} - 1\big)|C|^2 - \frac{\alpha_1|CD - DC|^2}{4\beta|C|^2} \tag{15}$$

and squaring (11) and using that $\operatorname{tr}(C) = 0$ yields

$$|B_0|^2 = -\frac{32(\alpha_1 + \alpha_2)^2[\operatorname{tr}(C^3)]^2}{9\beta^2|C|^6} + \frac{4(\alpha_1 + \alpha_2)^2}{\beta^2}\big(\frac{1}{2} - \frac{1}{n}\big) + \frac{\alpha_1^2|CD - DC|^2}{4\beta^2|C|^4}. \tag{16}$$

Using (10), (13), (14), (15) and (16) we finally get the following formula for $F(B_0)$:

$$F(B_0) = \Big[\nu + \frac{(\alpha_1+\alpha_2)^2}{\beta}\big(\frac{1}{n}-\frac{1}{2}\big)\Big]|C|^2 - \frac{\alpha_1^2|CD-DC|^2}{16\beta|C|^2} + \frac{\beta}{4}|C|^4 + \frac{2(\alpha_1+\alpha_2)^2[\operatorname{tr}(C^3)]^2}{3\beta|C|^4} \tag{17}$$

We claim next that

$$|CD-DC|^2 \le 2|C|^2|D|^2. \tag{18}$$

In dimension two, since C is symmetric trace free and D is antisymmetric, it is an easy calculation to check that equality holds in the above relation. In dimension three, we diagonalize again the matrix C and use that if O is an orthogonal matrix and F is an arbitrary matrix, then $|OF| = |FO| = |F|$. We write

$$|CD-DC| = |O\overline{C}O^tD - DO\overline{C}O^t| = |\overline{C}O^tDO - O^tDO\overline{C}| = |\overline{C}\widetilde{D} - \widetilde{D}\overline{C}|$$

where $\overline{C}$ is a trace free diagonal matrix and $\widetilde{D} = O^tDO$ is an antisymmetric matrix. Moreover, $|C| = |\overline{C}|$ and $|\widetilde{D}| = |D|$. Let

$$\overline{C} = \begin{pmatrix} a & 0 & 0\\ 0 & b & 0\\ 0 & 0 & c\end{pmatrix} \qquad \text{and} \qquad \widetilde{D} = \begin{pmatrix} 0 & x & y\\ -x & 0 & z\\ -y & -z & 0\end{pmatrix}.$$

We compute explicitly

$$|\overline{C}\widetilde{D} - \widetilde{D}\overline{C}|^2 = 2\big[x^2(a-b)^2 + y^2(a-c)^2 + z^2(b-c)^2\big] \le 2(x^2+y^2+z^2)\max\big[(a-b)^2,(a-c)^2,(b-c)^2\big].$$

We can assume without loss of generality that $a \ge b \ge 0$. Since $c = -a-b$ we infer that

$$\max\big[(a-b)^2,(a-c)^2,(b-c)^2\big] = (2a+b)^2 \le 2[a^2+b^2+(a+b)^2].$$

Consequently,

$$|CD-DC|^2 = |\overline{C}\widetilde{D} - \widetilde{D}\overline{C}|^2 \le 4(x^2+y^2+z^2)(a^2+b^2+c^2) = 2|\widetilde{D}|^2|\overline{C}|^2 = 2|C|^2|D|^2$$

which completely proves (18). Using now (18) in (17) gives

$$F(B_0) \ge \Big[\nu + \frac{(\alpha_1+\alpha_2)^2}{\beta}\big(\frac{1}{n}-\frac{1}{2}\big)\Big]|C|^2 - \frac{\alpha_1^2}{8\beta}|D|^2 + \frac{\beta}{4}|C|^4 + \frac{2(\alpha_1+\alpha_2)^2[\operatorname{tr}(C^3)]^2}{3\beta|C|^4} \tag{19}$$

On the other hand, we observe that the vector field $w = u - \widetilde{u}$ is divergence free, vanishes on the boundary, $C = \nabla w + (\nabla w)^t$ and $D = \nabla w - (\nabla w)^t$. Then

$$\begin{aligned}\int_\Omega |D|^2 &= \sum_{ij}\int_\Omega(\partial_i w_j - \partial_j w_i)^2 = 2\sum_{ij}\int_\Omega(\partial_i w_j)^2 - 2\sum_{ij}\int_\Omega \partial_i w_j \partial_j w_i\\ &= 2\sum_{ij}\int_\Omega(\partial_i w_j)^2 + 2\sum_{ij}\int_\Omega \partial_j\partial_i w_j w_i = 2\sum_{ij}\int_\Omega(\partial_i w_j)^2 - 2\sum_{ij}\int_\Omega \partial_j\partial_i w_j w_i = \int_\Omega |C|^2\end{aligned}$$

We deduce from the above relation and from (5) and (19) that

$$2\langle \mathcal{R}(u) - \mathcal{R}(\widetilde{u}), u - \widetilde{u}\rangle_{W^{-1,\frac{4}{3}},W_0^{1,4}} \ge \int_\Omega \Big[\nu + \frac{(\alpha_1+\alpha_2)^2}{\beta}\big(\frac{1}{n}-\frac{1}{2}\big) - \frac{\alpha_1^2}{8\beta}\Big]|C|^2 + \frac{\beta}{4}|C|^4 + \frac{2(\alpha_1+\alpha_2)^2[\operatorname{tr}(C^3)]^2}{3\beta|C|^4}$$

With the assumptions we have made on the coefficients $\alpha_1, \alpha_2, \nu, \beta$ the right-hand side of the above relation is obviously non-negative. The conclusion follows. $\square$

We prove next the following crucial lemma.

Lemma 5. *Let u and v be two vector fields belonging to $W^{1,4}_{0,\sigma}$. Then*

$$\sum_j \langle v_j A, \partial_j \nabla \widetilde{J}_\varepsilon u\rangle_{\mathcal{D}',\mathcal{D}} \to 0 \qquad \text{as} \quad \varepsilon \to 0,$$

where $A = A(u)$.

Even though the integral below is not convergent in the usual sense, we will use in the sequel the standard notation

$$\sum_j \langle v_j A, \partial_j \nabla \widetilde{J}_\varepsilon u\rangle_{\mathcal{D}',\mathcal{D}} = \int_\Omega v\cdot\nabla A : \nabla \widetilde{J}_\varepsilon u,$$

and we will also use the same notation for other similar terms.

Proof. We denote by I^ε the integral from the statement. For notational convenience, we treat the case $n=2$. It will be clear from the proof that in the case $n=3$ the argument is strictly similar, just the notation is different. Indeed, we will not use any Sobolev embedding, only the Hardy and Hölder inequalities which are dimension free. Let $u_\varepsilon = \widetilde{J}_\varepsilon u$. We write

$$\begin{aligned}
\int_\Omega v\cdot\nabla A : \nabla u_\varepsilon &= \int_\Omega v\cdot\nabla A : \nabla\nabla^\perp[\phi_\varepsilon * (h_\varepsilon\psi)]\\
&= \underbrace{\int_\Omega v\cdot\nabla A : J_\varepsilon \nabla u}_{I_1^\varepsilon} + \underbrace{\int_\Omega v\cdot\nabla A : [\phi_\varepsilon * (\psi\nabla\nabla^\perp h_\varepsilon)]}_{I_2^\varepsilon}\\
&\quad + \underbrace{\int_\Omega v\cdot\nabla A : [\phi_\varepsilon * (\nabla h_\varepsilon \otimes \nabla^\perp\psi)]}_{I_3^\varepsilon} + \underbrace{\int_\Omega v\cdot\nabla A : [\phi_\varepsilon * (\nabla^\perp h_\varepsilon \otimes \nabla\psi)]}_{I_4^\varepsilon}.
\end{aligned} \tag{20}$$

Let Γ_ε denote the ε-neighborhood of the boundary:

$$\Gamma_\varepsilon = \{x\in\Omega;\quad d(x,\partial\Omega)<\varepsilon\}.$$

Clearly $\mathrm{vol}(\Gamma_\varepsilon)\le C\varepsilon$, for some constant C. Given the localisation properties of ϕ_ε and h_ε we observe that the supports of the integrands in $I_2^\varepsilon, I_3^\varepsilon$ and I_4^ε are included in $\Gamma_{4\varepsilon}$. Next, for $p\in[1,4]$ we use the Holder inequality and the Hardy inequality given in Lemma 3 to write:

$$\|\nabla u\|_{L^p(\Gamma_{4\varepsilon})} \le \|\nabla u\|_{L^4}\|1\|_{L^{\frac{4p}{4-p}}(\Gamma_{4\varepsilon})} \le C\varepsilon^{\frac1p-\frac14}\|u\|_{W^{1,4}} \tag{21}$$

$$\|u\|_{L^p(\Gamma_{4\varepsilon})} \le C\varepsilon^{\frac1p-\frac14}\|u\|_{L^4(\Gamma_{4\varepsilon})} \le C\varepsilon^{\frac1p+\frac34}\|\frac{u}{d}\|_{L^4} \le C\varepsilon^{\frac1p+\frac34}\|u\|_{W^{1,4}} \tag{22}$$

$$\|\nabla\psi\|_{L^p(\Gamma_{4\varepsilon})} \le C\varepsilon^{\frac1p+\frac34}\|\psi\|_{W^{2,4}} \le C\varepsilon^{\frac1p+\frac34}\|u\|_{W^{1,4}} \tag{23}$$

$$\|\psi\|_{L^p(\Gamma_{4\varepsilon})} \le C\varepsilon^{\frac1p-\frac14}\|\psi\|_{L^4(\Gamma_{4\varepsilon})} \le C\varepsilon^{\frac1p+\frac74}\|\psi\|_{W^{2,4}} \le C\varepsilon^{\frac1p+\frac74}\|u\|_{W^{1,4}} \tag{24}$$

and similar relations hold true for v.

We integrate by parts using that u is divergence free, use the Holder and Young inequality, together with relations (22) for v, (21) and (24) to get that

$$\begin{aligned}
|I_2^\varepsilon| =& \Big|\int_{\Gamma_{4\varepsilon}} \{v\cdot[\nabla\phi_\varepsilon * (\nabla\nabla^\perp h_\varepsilon\psi)]\} : A\Big|\\
\le& \|v\|_{L^3(\Gamma_{4\varepsilon})}\|A\|_{L^3(\Gamma_{4\varepsilon})}\|\nabla\phi_\varepsilon * (\psi\nabla\nabla^\perp h_\varepsilon)\|_{L^3}\\
\le& \|v\|_{L^3(\Gamma_{4\varepsilon})}\|A\|_{L^3(\Gamma_{4\varepsilon})}\|\nabla\phi_\varepsilon\|_{L^1}\|\nabla\nabla^\perp h_\varepsilon\|_{L^\infty}\|\psi\|_{L^3(\Gamma_\varepsilon)}\\
\le& C\varepsilon^{\frac14}\|u\|_{W^{1,4}}^2\|v\|_{W^{1,4}}.
\end{aligned} \tag{25}$$

We used above that $\|\nabla\phi_\varepsilon\|_{L^1}\le C/\varepsilon$ and $\|\nabla\nabla^\perp h_\varepsilon\|_{L^\infty}\le C/\varepsilon^2$. Similarly, using also (23),

$$\begin{aligned}
|I_3^\varepsilon| =& \Big|\int_{\Gamma_{4\varepsilon}} \{v\cdot[\nabla\phi_\varepsilon * (\nabla h_\varepsilon\otimes\nabla^\perp\psi)]\} : A\Big|\\
\le& \|v\|_{L^3(\Gamma_{4\varepsilon})}\|A\|_{L^3(\Gamma_{4\varepsilon})}\|\nabla\phi_\varepsilon\|_{L^1}\|\nabla h_\varepsilon\|_{L^\infty}\|\nabla\psi\|_{L^3}\\
\le& C\varepsilon^{\frac14}\|u\|_{W^{1,4}}^2\|v\|_{W^{1,4}}
\end{aligned} \tag{26}$$

and also

$$|I_4^\varepsilon| \le C\varepsilon^{\frac14}\|u\|_{W^{1,4}}^2\|v\|_{W^{1,4}}. \tag{27}$$

We next deal with I_1^ε. For notational convenience, we extend u, v and A, with 0 outside Ω. We denote by $\widetilde{u}$, $\widetilde{v}$ and $\widetilde{A}$ the corresponding extension. Observe that $\widetilde{A} = A(\widetilde{u})$ in $\mathbb{R}^n$ and that $\widetilde{v}\cdot\nabla\widetilde{A}$

is simply the extension of $v \cdot \nabla A$ with 0 outside Ω (no jump occurs on $\partial\Omega$). This allows to have integrals on $\mathbb{R}^n$ instead of Ω while dealing with compactly supported functions. One has that

$$
\begin{aligned}
I_1^\varepsilon &= \int_{\mathbb{R}^n} \widetilde{v} \cdot \nabla \widetilde{A} : [\phi_\varepsilon * (h_\varepsilon \nabla \widetilde{u})] \\
&= \int_{\mathbb{R}^n} [\phi_\varepsilon * (\widetilde{v} \cdot \nabla \widetilde{A})] : (h_\varepsilon \nabla \widetilde{u}) \\
&= \int_{\mathbb{R}^n} [\widetilde{v} \cdot \nabla(\phi_\varepsilon * \widetilde{A})] : (h_\varepsilon \nabla \widetilde{u}) + \int_{\mathbb{R}^n} [\phi_\varepsilon * (\widetilde{v} \cdot \nabla \widetilde{A}) - \widetilde{v} \cdot \nabla(\phi_\varepsilon * \widetilde{A})] : (h_\varepsilon \nabla \widetilde{u}) \\
&= \underbrace{\int_{\mathbb{R}^n} \widetilde{v} \cdot \nabla (J_\varepsilon \widetilde{A}) : \nabla \widetilde{u}}_{I_{11}^\varepsilon} + \underbrace{\sum_i \int_{\mathbb{R}^n} \widetilde{v}_i [(\partial_i \phi_\varepsilon * \widetilde{A}) h_\varepsilon - \partial_i \phi_\varepsilon * (h_\varepsilon \widetilde{A})] : \nabla \widetilde{u}}_{I_{12}^\varepsilon} \\
&\qquad + \underbrace{\int_{\mathbb{R}^n} [\phi_\varepsilon * (\widetilde{v} \cdot \nabla \widetilde{A}) - \widetilde{v} \cdot \nabla(\phi_\varepsilon * \widetilde{A})] : (h_\varepsilon \nabla \widetilde{u})}_{I_{13}^\varepsilon}
\end{aligned} \tag{28}
$$

Next,

$$
\begin{aligned}
I_{11}^\varepsilon = \int_\Omega v \cdot \nabla (J_\varepsilon A) : \nabla u &= \int_\Omega v \cdot \nabla (J_\varepsilon \nabla u + J_\varepsilon (\nabla u)^t) : \nabla u \\
&= \int_\Omega v \cdot \nabla J_\varepsilon (\nabla u) : A = - \int_\Omega v \cdot \nabla A : J_\varepsilon (\nabla u) = -I_1^\varepsilon.
\end{aligned} \tag{29}
$$

On the other hand, due to the localisation properties of ϕ_ε and h_ε, we observe that

$$
\operatorname{supp}\big[(\partial_i \phi_\varepsilon * \widetilde{A}) h_\varepsilon - \partial_i \phi_\varepsilon * (h_\varepsilon \widetilde{A})\big] \subset \Gamma_{4\varepsilon},
$$

so one can deduce as in the estimates of $I_2^\varepsilon, I_2^\varepsilon$ and I_4^ε that

$$
|I_{12}^\varepsilon| \leq C \varepsilon^{\frac{1}{4}} \|u\|_{W^{1,4}}^2 \|v\|_{W^{1,4}}. \tag{30}
$$

Next, we bound

$$
|I_{13}^\varepsilon| \leq \|\nabla u\|_{L^2} \|\phi_\varepsilon * (\widetilde{v} \cdot \nabla \widetilde{A}) - \widetilde{v} \cdot \nabla (\phi_\varepsilon * \widetilde{A})\|_{L^2}.
$$

Since $\widetilde{v} \in W^{1,4}(\mathbb{R}^n)$ and $\widetilde{A} \in L^4(\mathbb{R}^n)$, according to [13, Lemma II.1] one has that

$$
\phi_\varepsilon * (\widetilde{v} \cdot \nabla \widetilde{A}) - \widetilde{v} \cdot \nabla (\phi_\varepsilon * \widetilde{A}) \xrightarrow{\varepsilon \to 0} 0 \quad \text{strongly in} \quad L^2
$$

so that

$$
I_{13}^\varepsilon \xrightarrow{\varepsilon \to 0} 0. \tag{31}
$$

Collecting relations (28), (29), (30) and (31) gives that

$$
I_1^\varepsilon = \frac{1}{2} I_{12}^\varepsilon + \frac{1}{2} I_{13}^\varepsilon \xrightarrow{\varepsilon \to 0} 0. \tag{32}
$$

We finally deduce from relations (20), (25), (26), (27) and (32) that

$$
I^\varepsilon \longrightarrow 0 \quad \text{as} \quad \varepsilon \to 0
$$

and this concludes the proof. □

3. Existence of H^1 solutions

Let $\varphi_n \in V$ be the classical basis of eigenfunctions of the Stokes operator $-\mathbb{P}\triangle$, where $\mathbb{P}$ is the Leray projector. That is

$$
-\triangle \varphi_n = \lambda_n \varphi_n + \nabla p_n, \qquad \varphi_n|_{\partial\Omega} = 0, \qquad \operatorname{div} \varphi_n = 0, \tag{33}
$$

the set $\{\varphi_n\}$ is an orthonormal basis of H, an orthogonal basis of V and the sequence $\{\lambda_n\}$ is increasing. Let Π_n be the L^2 orthogonal projection on the subspace spanned by the first n eigenfunctions

$\varphi_1, \ldots, \varphi_n$. We use the Galerkin procedure to construct an approximate system which will admit a solution. Let $u_n = \sum_{k=1}^{n} a_k^n \varphi_k$ be a solution of the following system of equations:

$$\langle u_n \cdot \nabla u_n - \alpha_1 \operatorname{div}(u_n \cdot \nabla A_n) + \mathcal{R}(u_n), \varphi_j \rangle = \langle f, \varphi_j \rangle_{W^{-1,\frac{4}{3}}, W_0^{1,4}}, \qquad \forall j \in \{1, \ldots, n\}. \tag{34}$$

We used the notation $A_n = A(u_n)$. We argue now that such a solution indeed exists. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be defined by $T(a)_j = \langle u \cdot \nabla u - \alpha_1 \operatorname{div}(u \cdot \nabla A) + \mathcal{R}(u) - f, \varphi_j \rangle$ for all $j \in \{1, \ldots, n\}$ where $u = \sum_{k=1}^{n} a_k \varphi_k$. We need to prove that there exists an $a \in \mathbb{R}^n$ such that $T(a) = 0$. Clearly T is continuous. We prove next that there exists M such that $T(a) \cdot a \geq 0$ for all $|a| \geq M$. The existence of a zero of T then follows from the classical Brouwer theorem, see for example [11, Lemma 7.2].

We observe that

$$\langle u \cdot \nabla u, u \rangle = \int_\Omega u \cdot \nabla u \cdot u = 0$$

and

$$\langle \operatorname{div}(u \cdot \nabla A), u \rangle = -\int_\Omega u \cdot \nabla A : \nabla u = -\frac{1}{2} \int_\Omega u \cdot \nabla A : A = 0$$

so

$$\begin{aligned} T(a) \cdot a = \langle u \cdot \nabla u - \alpha_1 \operatorname{div}(u \cdot \nabla A) + \mathcal{R}(u) - f, u \rangle = \langle \mathcal{R}(u), u \rangle - \langle f, u \rangle_{W^{-1,\frac{4}{3}}, W_0^{1,4}} \\ \geq \langle \mathcal{R}(u), u \rangle - \|f\|_{W^{-1,\frac{4}{3}}} \|u\|_{W^{1,4}}. \end{aligned}$$

With similar calculations as in Lemma 4, one has that

$$\langle \mathcal{R}(u), u \rangle = \frac{1}{2} \int_\Omega [\nu |A|^2 + (\alpha_1 + \alpha_2) \operatorname{tr}(A^3) + \beta |A|^4].$$

In dimension two, since u is divergence free we have that $\operatorname{tr}(A^3) = 0$. Moreover, by the Korn inequality, $\|A\|_{L^4} \simeq \|u\|_{W^{1,4}}$. We infer that

$$T(a) \cdot a \geq \frac{\beta}{2} \|A\|_{L^4}^4 - \|f\|_{W^{-1,\frac{4}{3}}} \|u\|_{W^{1,4}} \geq \|u\|_{W^{1,4}} (C \|u\|_{W^{1,4}}^3 - \|f\|_{W^{-1,\frac{4}{3}}}) \geq \|u\|_{W^{1,4}} (C \|u\|_V^3 - \|f\|_{W^{-1,\frac{4}{3}}}),$$

where we used the embedding $W^{1,4} \hookrightarrow V$. Since $\varphi_1, \ldots, \varphi_n$ are orthogonal in V, clearly $\|u\|_V \to \infty$ as $|a| \to \infty$. Therefore, there exists M such that $\|u\|_V \geq (\|f\|_{W^{-1,\frac{4}{3}}} / C)^{\frac{1}{3}}$ for all $|a| \geq M$. Obviously $T(a) \cdot a \geq 0$ for all $|a| \geq M$.

In dimension three, we know from [14, Lemma 3] that if $\nu, \beta \geq 0$ and $|\alpha_1 + \alpha_2| \leq \sqrt{24 \nu \beta}$ then $\nu |A|^2 + (\alpha_1 + \alpha_2) \operatorname{tr}(A^3) + \beta |A|^4 \geq 0$. Since $n = 3$, here we assumed the more restrictive assumption $\nu, \beta \geq 0$ and $|\alpha_1 + \alpha_2| \leq \sqrt{6 \nu \beta}$. Therefore,

$$\langle \mathcal{R}(u), u \rangle = \frac{1}{2} \int_\Omega [\nu |A|^2 + (\alpha_1 + \alpha_2) \operatorname{tr}(A^3) + \frac{\beta}{4} |A|^4] + \frac{3\beta}{8} \int_\Omega |A|^4 \geq \frac{3\beta}{8} \int_\Omega |A|^4.$$

We deduce as in the bidimensional case that there exists $M > 0$ such that $T(a) \cdot a \geq 0$ for all $|a| \geq M$. This concludes the proof of the existence of the approximate solution u_n.

Multiplying (34) by a_j^n and summing over j we obtain that

$$\begin{aligned} 2\langle \mathcal{R}(u_n), u_n \rangle = \int_\Omega [\nu |A_n|^2 + (\alpha_1 + \alpha_2) \operatorname{tr}(A_n^3) + \beta |A_n|^4] = 2\langle f, u_n \rangle_{W^{-1,\frac{4}{3}}, W_0^{1,4}} \\ \leq 2 \|f\|_{W^{-1,\frac{4}{3}}} \|u_n\|_{W^{1,4}}. \end{aligned} \tag{35}$$

According to the remarks above, the left-hand side is bounded from below as follows

$$\int_\Omega [\nu |A_n|^2 + (\alpha_1 + \alpha_2) \operatorname{tr}(A_n^3) + \beta |A_n|^4] \geq \frac{3\beta}{8} \int_\Omega |A_n|^4 \geq C_1 \|u_n\|_{W^{1,4}}^4. \tag{36}$$

We infer from (35) and (36) that $\|u_n\|_{W^{1,4}} \le (2\|f\|_{W^{-1,\frac{4}{3}}}/C_1)^{\frac{1}{3}}$. The sequence u_n is therefore bounded in $W^{1,4}$. By the compact embedding $W^{1,4} \hookrightarrow L^2$, we finally deduce that there is a subsequence, again denoted by u_n, such that

$$u_n \rightharpoonup u \quad \text{in} \quad W^{1,4} \quad \text{weakly} \tag{37}$$

and

$$u_n \to u \quad \text{in} \quad L^2 \quad \text{strongly} \tag{38}$$

as $n \to \infty$. Moreover, one also has that $\mathcal{R}(u_n)$ is bounded in $W^{-1,\frac{4}{3}}$, so there exists $\zeta \in W^{-1,\frac{4}{3}}$ such that

$$\mathcal{R}(u_n) \rightharpoonup \zeta \quad \text{in} \quad W^{-1,\frac{4}{3}} \quad \text{weakly.} \tag{39}$$

We fix j and pass to the limit as $n \to \infty$ in (34). We clearly have from (37) and (38) that

$$\langle u_n \cdot \nabla u_n, \varphi_j \rangle \overset{n\to\infty}{\longrightarrow} \langle u \cdot \nabla u, \varphi_j \rangle,$$
$$\langle \operatorname{div}(u_n \cdot \nabla A_n), \varphi_j \rangle = \sum_i \int_\Omega u_{n,i} A_n : \partial_i \nabla \varphi_j \overset{n\to\infty}{\longrightarrow} \sum_i \int_\Omega u_i A : \partial_i \nabla \varphi_j$$

and, in view of (39),

$$\langle \mathcal{R}(u_n), \varphi_j \rangle \overset{n\to\infty}{\longrightarrow} \langle \zeta, \varphi_j \rangle_{W^{-1,\frac{4}{3}}, W_0^{1,4}}.$$

Therefore

$$\langle u \cdot \nabla u + \zeta - f, \varphi_j \rangle_{W^{-1,\frac{4}{3}}, W_0^{1,4}} - \alpha_1 \sum_i \int_\Omega u_i A : \partial_i \nabla \varphi_j = 0 \qquad \forall j.$$

It is well-known that the set of eigenfunctions φ_j form a complete system of $\mathcal{D}(-\mathbb{P}\triangle) = H^2 \cap H^1_{0,\sigma}$, see for example [11, Chapter 4]. Therefore, the equality above holds true with φ_j replaced with any function in $H^2 \cap H^1_{0,\sigma}$. In particular, it holds true with φ_j replaced by any element of $C^\infty_{0,\sigma}$:

$$\langle u \cdot \nabla u - \alpha_1 \operatorname{div}(u \cdot \nabla A) + \zeta - f, \varphi \rangle_{\mathcal{D}', \mathcal{D}} = 0 \qquad \forall \varphi \in C^\infty_{0,\sigma}.$$

Equivalently, there exists some scalar function p such that

$$u \cdot \nabla u - \alpha_1 \operatorname{div}(u \cdot \nabla A) + \zeta = f - \nabla p \tag{40}$$

in the sense of distributions. The rest of this proof consists in showing that $\zeta = \mathcal{R}(u)$. This equality is proved with monotonicity arguments as follows. We will show that

$$\langle \zeta - \mathcal{R}(\varphi), u - \varphi \rangle_{W^{-1,\frac{4}{3}}, W_0^{1,4}} \ge 0 \qquad \forall \varphi \in C^\infty_{0,\sigma}. \tag{41}$$

Let us assume for the moment that the above relation is true and let us conclude the proof. We will use an idea known under the name of "Minty argument". We observe by a density argument that (41) holds true for any vector field $\varphi \in W^{1,4}_{0,\sigma}$. Choose now $\varphi = u + \lambda\psi$, where $\psi \in W^{1,4}_{0,\sigma}$ is arbitrary and $\lambda > 0$. We obtain then

$$\langle \zeta - \mathcal{R}(u + \lambda\psi), \psi \rangle_{W^{-1,\frac{4}{3}}, W_0^{1,4}} \le 0$$

From the explicit expression of $\mathcal{R}$ given in relation (3), we observe that the application $\mathbb{R} \ni \lambda \mapsto \langle \mathcal{R}(u + \lambda\psi), \psi \rangle_{W^{-1,\frac{4}{3}}, W_0^{1,4}}$ is polynomial, therefore continuous. Letting $\lambda \to 0$ in the above inequality yields

$$\langle \zeta - \mathcal{R}(u), \psi \rangle_{W^{-1,\frac{4}{3}}, W_0^{1,4}} \le 0$$

Changing ψ into $-\psi$ we infer that equality must hold above. This clearly implies the desired equality $\zeta = \mathcal{R}(u)$.

It remains to prove (41). We write the following computation

$$\langle \zeta - \mathcal{R}(\varphi), u - \varphi \rangle_{W^{-1,\frac{4}{3}},W_0^{1,4}} = \underbrace{\langle \mathcal{R}(u_n) - \mathcal{R}(\varphi), u_n - \varphi \rangle_{W^{-1,\frac{4}{3}},W_0^{1,4}}}_{I_1} + \underbrace{\langle \mathcal{R}(u_n) - \zeta, \varphi \rangle_{W^{-1,\frac{4}{3}},W_0^{1,4}}}_{I_2}$$
$$+ \underbrace{\langle \mathcal{R}(\varphi), u_n - u \rangle_{W^{-1,\frac{4}{3}},W_0^{1,4}}}_{I_3} + \underbrace{\langle \zeta, u \rangle_{W^{-1,\frac{4}{3}},W_0^{1,4}} - \langle \mathcal{R}(u_n), u_n \rangle_{W^{-1,\frac{4}{3}},W_0^{1,4}}}_{I_4} \tag{42}$$

We observe first that $\mathcal{R}$ being monotone, one has that $I_1 \geq 0$. From the definition of ζ we infer that $I_2 \to 0$ as $n \to \infty$. Next, by (37),

$$I_3 = \langle \mathcal{R}(\varphi), u_n - u \rangle_{W^{-1,\frac{4}{3}},W_0^{1,4}} \to 0 \qquad \text{as } n \to \infty.$$

Recall now relation (35):

$$\langle \mathcal{R}(u_n), u_n \rangle_{W^{-1,\frac{4}{3}},W_0^{1,4}} = \langle f, u_n \rangle_{W^{-1,\frac{4}{3}},W_0^{1,4}}. \tag{43}$$

We next show that one can multiply (40) by u. To do that, we first multiply by $u_\varepsilon = \widetilde{J}_\varepsilon u$ and integrate to obtain

$$\int_\Omega u \cdot \nabla u \cdot u_\varepsilon - \alpha_1 \int_\Omega \sum_{ij} u_i A : \partial_i \nabla u_\varepsilon + \langle \zeta, u_\varepsilon \rangle = \langle f, u_\varepsilon \rangle_{W^{-1,\frac{4}{3}},W_0^{1,4}}. \tag{44}$$

Since $\zeta \in W^{-1,\frac{4}{3}}$ and $u_\varepsilon \to u$ in $W^{1,4}$ strongly, we see that

$$\langle \zeta, u_\varepsilon \rangle_{W^{-1,\frac{4}{3}},W_0^{1,4}} \xrightarrow{\varepsilon \to 0} \langle \zeta, u \rangle_{W^{-1,\frac{4}{3}},W_0^{1,4}},$$

and

$$\int_\Omega u \cdot \nabla u \cdot u_\varepsilon \xrightarrow{\varepsilon \to 0} \int_\Omega u \cdot \nabla u \cdot u = 0.$$

Furthermore, by Lemma 5 one has that

$$\int_\Omega \sum_{ij} u_i A : \partial_i \nabla u_\varepsilon \xrightarrow{\varepsilon \to 0} 0.$$

Therefore, taking the limit $\varepsilon \to 0$ in (44) results in

$$\langle \zeta, u \rangle_{W^{-1,\frac{4}{3}},W_0^{1,4}} = \langle f, u \rangle_{W^{-1,\frac{4}{3}},W_0^{1,4}}$$

so, in view of (43),

$$I_4 = \langle f, u - u_n \rangle_{W^{-1,\frac{4}{3}},W_0^{1,4}} \to 0 \qquad \text{as } n \to \infty.$$

Taking the limit $n \to \infty$ in (42) implies the desired property (41) and concludes the proof of the existence of a solution in the sense of distributions, except for the validity of the energy equality (2). We already proved that (40) can be multiplied by u, that the first two terms vanish when we do that and also that $\zeta = \mathcal{R}(u)$. Therefore, one can multiply (40) by u to obtain that

$$\langle f, u \rangle_{W^{-1,\frac{4}{3}},W_0^{1,4}} = \langle \mathcal{R}(u), u \rangle_{W^{-1,\frac{4}{3}},W_0^{1,4}} = \frac{1}{2} \int_\Omega [\nu |A|^2 + (\alpha_1 + \alpha_2) \operatorname{tr}(A^3) + \beta |A|^4].$$

This completes the proof of Theorem 1.

4. Weak-strong uniqueness

In this section we prove Theorem 2. Let $u \in W_0^{1,4}$ and $\widetilde{u} \in W_0^{1,4} \cap W^{2,p}$ be two solutions of system (1). We use the notations

$$w = u - \widetilde{u}, \quad A = A(u), \quad \widetilde{A} = A(\widetilde{u}), \quad L = L(u), \quad \widetilde{L} = L(\widetilde{u}).$$

We denote by K a generic constant that may depend on the domain Ω and on p. The difference w verifies the equation

$$-\alpha_1 \operatorname{div}[u \cdot \nabla A(w)] - \alpha_1 \operatorname{div}[w \cdot \nabla \widetilde{A}] + (u \cdot \nabla u - \widetilde{u} \cdot \nabla \widetilde{u}) + \mathcal{R}(u) - \mathcal{R}(\widetilde{u}) = -\nabla(p - \widetilde{p})$$

in the sense of distributions. We multiply by $w_\varepsilon = \widetilde{J}_\varepsilon w$ and integrate in space to obtain

$$(45)\quad -\alpha_1 \sum_j \langle u_j A(w), \partial_j \nabla w_\varepsilon \rangle_{\mathcal{D}',\mathcal{D}} + \alpha_1 \int_\Omega w \cdot \nabla \widetilde{A} : \nabla w_\varepsilon + \int_\Omega (u \cdot \nabla u - \widetilde{u} \cdot \nabla \widetilde{u}) \cdot w_\varepsilon \\ + \langle \mathcal{R}(u) - \mathcal{R}(\widetilde{u}), w_\varepsilon \rangle_{W^{-1,\frac{4}{3}}, W_0^{1,4}} = 0.$$

By Lemma 5, the first term above converges to 0 as $\varepsilon \to 0$. Given that $w_\varepsilon \to w$ strongly in $W^{1,4}$, it is easy to pass to the limit in (45) to obtain that

$$(46)\qquad \langle \mathcal{R}(u) - \mathcal{R}(\widetilde{u}), w \rangle_{W^{-1,\frac{4}{3}}, W_0^{1,4}} = -\int_\Omega (u \cdot \nabla u - \widetilde{u} \cdot \nabla \widetilde{u}) \cdot w - \alpha_1 \int_\Omega w \cdot \nabla \widetilde{A} : A(w).$$

An easy integration by parts shows that

$$(47)\qquad \Big| \int_\Omega (u \cdot \nabla u - \widetilde{u} \cdot \nabla \widetilde{u}) \cdot w \Big| = \Big| \int_\Omega w \cdot \nabla w \cdot \widetilde{u} \Big| \le K \|w\|_{L^6} \|\nabla w\|_{L^2} \|\widetilde{u}\|_{L^3} \le K \|w\|_{H^1}^2 \|\widetilde{u}\|_{W^{2,p}},$$

where the value of p is given in the statement of Theorem 2. We used above the Sobolev embeddings $H^1 \hookrightarrow L^6$ and $W^{2,p} \hookrightarrow L^3$. Similarly, if the space dimension is three we bound

$$(48)\qquad \Big| \int_\Omega w \cdot \nabla \widetilde{A} : A(w) \Big| \le K \|w\|_{L^6} \|\nabla w\|_{L^2} \|\nabla \widetilde{A}\|_{L^3} \le K \|w\|_{H^1}^2 \|\widetilde{u}\|_{W^{2,3}} \qquad \text{if } n = 3,$$

while in dimension two we write

$$(49)\qquad \Big| \int_\Omega w \cdot \nabla \widetilde{A} : A(w) \Big| \le K \|w\|_{L^q} \|\nabla w\|_{L^2} \|\nabla \widetilde{A}\|_{L^p} \le K \|w\|_{H^1}^2 \|\widetilde{u}\|_{W^{2,p}} \qquad \text{if } n = 2,$$

where $q = 2p/(p-2)$.

Since the inequalities assumed on the material coefficients are strict, there exists $\varepsilon_0 > 0$ such that if we replace ν by $\nu - \varepsilon_0$, then these inequalities still hold true. Therefore, by Lemma 4, the operator $\mathcal{R}(u) + \varepsilon_0 \triangle u$ is still monotone so that

$$(50)\quad \langle \mathcal{R}(u) - \mathcal{R}(\widetilde{u}), w \rangle_{W^{-1,\frac{4}{3}}, W_0^{1,4}} = \langle \mathcal{R}(u) + \varepsilon_0 \triangle u - (\mathcal{R}(\widetilde{u}) + \varepsilon_0 \triangle \widetilde{u}), w \rangle_{W^{-1,\frac{4}{3}}, W_0^{1,4}} - \varepsilon_0 \langle \triangle w, w \rangle \\ \ge \frac{\varepsilon_0}{2} \|A(w)\|_{L^2}^2 \ge \frac{\varepsilon_0}{K} \|w\|_{H^1}^2.$$

Putting together relations (46)–(50), we finally get that

$$\varepsilon_0 \|w\|_{H^1}^2 \le K \|w\|_{H^1}^2 \|\widetilde{u}\|_{W^{2,p}},$$

which clearly implies that $w = 0$ provided that $\|\widetilde{u}\|_{W^{2,p}} < \varepsilon_0 / K$.

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